SOME CONDITIONS FOR MANIFOLDS TO BE LOCALLY FLAT[1]

BY

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Consider manifolds $M \subset N$ and a subset X of M, and assume that $M-X$ and X are locally nice in N. The general question considered in this paper is "What conditions on X imply that M is nice in N?" Without mentioning the case when N is three dimensional, this question has been considered before by Cantrell-Edwards in [9], by Cantrell in [6] and [7], by Edwards in [11], by Bryant in [5], by Lacher in [14], and elsewhere. However, in each of the above references the author restricts himself by either assuming that M lies in the trivial range or by assuming that X is a single point. The conditions derived in this paper make no dimensional restriction on M and assume only that $X \times [0, 1]$ lies in the trivial range.

The first three sections are devoted to studying embeddings of polyhedra into a manifold (in the trivial range). The polyhedra are allowed to intersect the boundary of the manifold. The results on embeddings in the trivial range constitute a major step in the proof of the main result of this paper (Theorem 4.2). The fourth and fifth sections derive some conditions for M to be nice in N when X lies in the boundary of M. The last section extends these results to the case when X lies in the interior of M, modulo a certain conjecture.

0. **Definitions and notations.** R^n is euclidean n-space, B^n is the closed unit ball in R^n, and S^n is the one-point compactification of R^n. S^n is triangulated so that R^n and B^n inherit their triangulations from S^n. When $m<n$, we identify R^m with $R^m \times 0 \subset R^n$. Thus we have $R^m \subset R^n \subset S^n$ and $B^m \subset B^n \subset S^n$ for $m<n$. An *n-cell* (*n-sphere*, *open n-cell*) is a space homeomorphic to B^n (resp. S^n, resp. R^n).

An *n-manifold* is a space N such that each point of N has a neighborhood whose closure is an n-cell; the *interior* of N (denoted by Int N) is the set of points of N which have open n-cell neighborhoods in N; the *boundary* of N (denoted by Bd N) is the complement of N-Int N of Int N.

Let M and N be manifolds of dimension m and n, respectively, with $M \subset \text{Int } N$. M is said to be *locally flat in N at the point* $x \in \text{Int } M$ if x has a neighborhood U in N such that $(U, U \cap M) \approx (R^n, R^m)$; i.e., the pairs $(U, U \cap M)$ and (R^n, R^m)

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are homeomorphic. M is *locally flat in N at the point* $x \in \text{Bd } M$ if x has a neighborhood U in N such that $(U, U \cap M) \approx (R^n, R^{m-1} \times [0, \infty))$. M is *locally flat in N* if M is locally flat at each point of M. In the special case when M is an m-cell and N is either R^n or S^n, we say that M is *flat* in N if $(N, M) \approx (N, B^m)$.

In this paper, a polyhedron or complex will be understood to be finite unless otherwise stated. A *combinatorial n-manifold* is an n-manifold N which has a locally finite triangulation in which the link of each vertex is either a combinatorial $(n-1)$-sphere or a combinatorial $(n-1)$-ball.

1. **Embeddings which intersect the boundary.** For the first two sections let the following be fixed: a (finite) k-complex K; a compact combinatorial n-manifold N; a closed subset A of K; and an embedding ϕ of A into Bd N.

Let Φ be any set of embeddings $f: K \to N$ such that $f \mid A = \phi$ and $f \mid K - A$ is an embedding of $K - A$ into Int N. Let d denote (ambiguously) a fixed metric for N and the uniform metric on Φ induced by d on N.

DEFINITION. Let X be a closed subset of N, $\varepsilon > 0$. An *ε-push h* of $(N, X; \text{Bd } N)$ is a homeomorphism of N such that

(1) h is an ε-homeomorphism of N onto itself; i.e., $d(x, h(x)) < \varepsilon$ for all x in N;

(2) h is the identity outside of the ε-neighborhood of X;

(3) h is the identity on Bd N; and

(4) h is isotopic to the identity on N through homeomorphisms satisfying (1), (2), and (3) above.

DEFINITION. A subset F of Φ is called *solvable* provided that for each $\varepsilon > 0$ there is a $\delta = \delta(F, \varepsilon) > 0$ such that if $f, g \in F$ and $d(f, g) < \delta$ then there is an ε-push h of $(N, f(K); \text{Bd } N)$ such that $hf = g$.

Notice that the above definition of solvability is a slight modification of that in [12]; however, the proofs of Theorems 4.3 and 4.4 of [12] need virtually no modification to yield the two lemmas below.

LEMMA 1.1. *Let $F \subset \Phi$, and suppose that for each $f \in \Phi$ and $\varepsilon > 0$ there is an ε-push h of $(N, f(K); \text{Bd } N)$ such that $hf \in F$. If F is solvable then so is Φ.*

LEMMA 1.2. *Let F_1 and F_2 be subsets of Φ, and suppose that each of F_1 and F_2 is dense in Φ. If F_1 and F_2 are solvable then $F_1 \cup F_2$ is solvable; in fact, given $\varepsilon > 0$, one may take $\delta(F_1 \cup F_2, \varepsilon)$ to be the minimum of $\{\delta(F_1, \varepsilon/6), \delta(F_2, \varepsilon/6)\}$.*

The next definition and lemma are based on the approximation techniques used by Bryant in [5].

DEFINITION. A subset F of Φ is called *weakly solvable* provided that for each $\varepsilon > 0$ there is a $\delta = \delta_w(F, \varepsilon) > 0$ such that if $f, g \in F$, $d(f, g) < \delta$, and f and g agree on a neighborhood of A in K, then there is an ε-push h of $(N, f(K); \text{Bd } N)$ such that $hf = g$.

Lemma 1.3. *Let F be any subset of Φ, and for each $g \in F$ let $F_g = \{f \in F : f$ agrees with g on a neighborhood of A in $K\}$. If F_g is dense in F for each $g \in F$, and if F is weakly solvable, then F is solvable.*

Proof. Let $\varepsilon > 0$, and let $\delta = \delta_w(F, \varepsilon/6)$ be given for $\varepsilon/6$ by the weak solvability of F. Suppose that $f, g \in F$ and that $d(f, g) < \delta$. Note that F_f and F_g are solvable dense subsets of F, and in fact $\delta(F_f, \varepsilon') = \delta(F_g, \varepsilon') = \delta_w(F, \varepsilon')$ for each $\varepsilon' > 0$. Hence $F_f \cup F_g$ is solvable by Lemma 1.2, and $\delta(F_f \cup F_g, \varepsilon)$ may be chosen to be $\min\{\delta(F_f, \varepsilon/6), \delta(F_g, \varepsilon/6)\} = \delta_w(F, \varepsilon/6) = \delta$. Therefore, since both f and g belong to $F_f \cup F_g$ and $d(f, g) < \delta$, there is an ε-push h of $(N, f(K);\ \mathrm{Bd}\, N)$ such that $hf = g$. Hence F is solvable.

2. **Solvability of locally tame embeddings in the trivial range.** The following definition of local tameness is the same as that used by Gluck in [12].

Definition. Let X be a locally finite polyhedron topologically embedded in the manifold M. X is said to be *locally tame* (with respect to the triangulation f) if there exist a locally finite complex L and a homeomorphism $f: L \approx X$ which satisfy the following condition: given a point x of L there is a neighborhood U of $f(x)$ in M and a triangulation of U as a combinatorial manifold with respect to which $|f^{-1}(U)f$ is piecewise linear.

An embedding f of a locally finite complex L into a manifold M is called *locally tame* if $f(L)$ is locally tame in M.

Keeping K, N, A, and ϕ fixed as in §1, we make the following additional assumptions:

$$\Phi = \{f: K \to N \mid f \mid A = \phi, \text{ and } f \mid K - A \text{ is a locally tame embedding } K - A \text{ into } \mathrm{Int}\, N\}.$$

$F = \{f \in \Phi \mid f$ is piecewise linear on $K - A\}$, and for each $g \in F$,
$F_g = \{f \in F \mid f$ agrees with g on a neighborhood of A in $K\}$.

The remainder of this section is devoted to the proof of the following theorem.

Theorem 2.1. *If $n \geqq 2k+2$ then Φ is solvable.*

The proof is given in several steps, each of which has essentially been done in the literature. Assume $n \geqq 2k+2$.

Lemma 2.2. *Given $\varepsilon > 0$ and $f \in \Phi$, there is an ε-push h of $(N, f(K);\ \mathrm{Bd}\, N)$ such that $hf \in F$.*

Proof. The proof follows easily by applying Theorem 9.1 of [12] an infinite number of times.

Lemma (Bing-Kister). *Suppose that K_1 and K_2 are two closed locally finite k-complexes in R^n and that h is a piecewise linear homeomorphism of K_1 onto K_2 that* *does not move any point as far as* ε, $n \geqq 2k+2$. *Suppose further that L is a subcomplex of* K_1 *such that* $(K_1-L)^-$ *is finite and* $h \mid L = \text{identity}$. *Then there is an isotopy* h_t $(0 \leqq t \leqq 1)$ *of* R^n *onto itself such that*

(i) $h_0 = \text{identity}$,

(ii) $h_1 \mid K_1 = h$,

(iii) *each* h_t *is piecewise linear on* R^n *and is the identity on L and outside the* ε*-neighborhood of* $K_1 - L$,

(iv) *each point of* R^n *moves along a polygonal path of length less than* ε.

The proof of the Bing-Kister lemma is essentially the same as the proof of Theorem 5.5 of [1].

Moreover, the proof of Lemma 2 of [13] can be used to prove the following lemma, substituting the Bing-Kister lemma for Proposition 1 of [13].

Lemma (Homma). *For any* $\varepsilon > 0$ *there is a* $\delta = \delta(N, \varepsilon) > 0$ *such that if* $\tilde{K}$ *is a k-complex,* $n \geqq 2k+2$, *and* $f, g: \tilde{K} \to N$ *are embeddings which satisfy*

(1) *f and g agree on a neighborhood of* $\tilde{A} = f^{-1}(\text{Bd } N)$ *in* $\tilde{K}$,

(2) *f and g are piecewise linear embeddings of* $\tilde{K} - \tilde{A}$ *into* Int N, *and*

(3) $d(f, g) < \delta$,

then there is a piecewise linear ε*-push h of* $(N, f(\tilde{K}); \text{Bd } N)$ *such that* $hf = g$; *moreover, hf may be taken to agree with f on some neighborhood of* $\tilde{A}$ *in* $\tilde{K}$.

Lemma 2.3. *F is weakly solvable.*

Proof. This follows immediately from the Homma lemma above.

Lemma 2.4. *For each* $g \in F$, F_g *is dense in F.*

Proof. Let $f, g \in F$; we will approximate f by a member of F_g. To do this let $\varepsilon > 0$ be given, and let $\delta = \delta(N, \varepsilon)$ be given by the Homma lemma.

Choose subcomplexes L_1 and L_2 of K (taking subdivisions if necessary) which satisfy the following conditions (where L_i° and L_i° denote the interior and boundary of L_i as subspaces of K).

(1) $A \subset L_1^\circ \subset L_1 \subset L_2^\circ$,

(2) $d(f|L_2, g|L_2) < \delta$,

(3) $g(L_1) \cap f(\dot{L}_2) = \varnothing$.

Define embeddings $\bar{f}$ and $\bar{g}$ of $L_1 \cup \dot{L}_2$ into N by letting $\bar{f}|L_1 = \bar{g}(L_1) = g|L_1$, $\bar{f}|\dot{L}_2 = f|\dot{L}_2$, and $\bar{g}|\dot{L}_2 = g|\dot{L}_2$. Thus $\bar{f}$ and $\bar{g}$ both agree with g on L_1, but agree with f and g respectively on $\dot{L}_2$. Conditions (1), (2), and (3) above, together with the Homma lemma, show that there is a piecewise linear ε-push h of

$$(N, \bar{g}(L_1 \cup \dot{L}_2); \text{Bd } N)$$

such that $h\bar{g} = \bar{f}$ and such that $h\bar{g}$ agrees with g on a neighborhood of A in K.

Now define f' on K by

$$f' = hg \quad \text{on } L_2,$$
$$= f \quad \text{on } \overline{K-L_2}.$$

f' is a mapping of K into N such that $f'|L_1 = g|L_1$, $f'|L_2$ is an embedding, $f' \mid K-A$ is piecewise linear, and $d(f', f) < \varepsilon + \delta \leqq 2\varepsilon$. A general position argument completes the proof.

Proof of Theorem 2.1. Lemmas 1.3, 2.3, and 2.4 show that F is solvable. But then Lemmas 1.1 and 2.2 show that Φ is solvable.

3. **An application.**

THEOREM 3.1. *Let N be a compact combinatorial n-manifold, K a k-complex, $n \geqq 2k+2$. Suppose that $f, g\colon K \to N$ are embeddings, $A = f^{-1}(\text{Bd } N)$, and $f|A = g|A$. If $f \mid K-A$ and $g \mid K-A$ are locally tame embeddings of $K-A$ into* Int N, *and if f and g are homotopic through maps $f_t\colon K \to N$ $(0 \leqq t \leqq 1)$ such that $f_t|A = f|A$ and $f_t(K-A) \subset N - f(A)$ for each t, then f and g are ambient isotopic leaving* Bd N *fixed.*

Proof. First observe that we may assume that $f_t(K-A) \subset \text{Int } N$ for each t. The reason for this is that there is a homotopy f'_t $(0 \leqq t \leqq 1)$ of N such that $f'_0 = f'_1 = \text{identity}$, $f'_t \mid f(A) = \text{identity}$ for each t, and $f'_t(N - f(A)) \subset \text{Int } N$ for $0 < t < 1$; f'_t $(0 \leqq t \leqq 1)$ may be constructed by pushing Bd $N - f(A)$ slightly into a collar for Bd N. Then the homotopy $f'_t f_t$ $(0 \leqq t \leqq 1)$ has the desired properties.

Let Φ be the set of embeddings of K into N which agree with $\phi = f \mid A$ on A and are locally tame embeddings of $K-A$ into Int N. By Theorem 2.1, Φ is solvable. (Note that f and g are in Φ.) Let $\delta = \delta(\Phi, 1) > 0$ be given by the solvability of Φ for $\varepsilon = 1$. Choose a finite sequence $t_0 = 0 < t_1 < \cdots < t_r = 1$ such that $d(f_{t_i}, f_{t_{i+1}}) < \delta/3$ for $i = 0, \ldots, r-1$. By a simplicial approximation and general position argument there are members $g_1, \ldots, g_{r-1}$ of Φ such that $d(g_i, f_{t_i}) < \delta/3$ for $i = 1, \ldots, r-1$. Letting $g_0 = f$ and $g_r = g$, we have $d(g_i, g_{i+1}) < \delta$ for $i = 0, \ldots, r-1$; hence there are 1-pushes h_i of $(N, g_{i-1}(K);\ \text{Bd } N)$ such that $h_i g_{i-1} = g_i$, $i = 1, \ldots, r$. Define $h = h_r \circ \cdots \circ h_1$; then $hf = g$, $h \mid \text{Bd } N = \text{identity}$ and h is isotopic to the identity through homeomorphisms which are the identity on Bd N. This completes the proof.

The following corollary is needed for the main result in §4.

COROLLARY 3.2. *Let Q be an n-cell, K a k-complex, $n \geqq 2k+2$, and let A be a closed subset of K. If $f, g\colon K \to Q$ are embeddings such that $f|A = g|A$ maps A into* Bd Q *and such that $f \mid K-A$ and $g \mid K-A$ are locally tame embeddings of $K-A$ into* Int Q, *then f and g are ambient isotopic leaving* Bd Q *fixed.*

Proof. It is clear that there is a homtopy f_t $(0 \leqq t \leqq 1)$ between f and g which satisfies the hypothesis of Theorem 3.1, because Q can be embedded in R^n as a convex set.

4. **Taming a cell at its boundary.** Before stating the main result of this paper, we will prove the following theorem.

THEOREM 4.1. *Let D be an m-cell in S^n, $n \geqq 4$, $m<n$, let X be a closed set in* Bd D, *and assume the following conditions*:

(1) *$D-X$ is locally flat in S^n*;

(2) *X is cellular in* Bd D; *and*

(3) *X is cellular in S^n.*

Then there is an embedding $\phi: B^n \to S^n$ such that $\phi(B^m)=D$ and $\phi(\text{Bd } B^n)-X$ is locally flat in S^n.

Proof. (For the definition of cellularity and basic facts, see [3].) Since X is cellular in S^n, there is a mapping π of S^n onto itself such that X is the only (nondegenerate) inverse set of π; moreover, since X is cellular in Bd D, $\pi(D)$ is an m-cell. Since $\pi(D)$ is locally flat at each point other than $\pi(X) \in \text{Bd } \pi(D)$, we may assume that $\pi(D)=B^m$ by Corollary 2.4 of [14]. Let $p=\pi(X)$, and let

$$g = \pi^{-1} \mid S^n-\{p\}.$$

The homeomorphism g takes $S^n-\{p\}$ onto S^n-X, and $B^m-\{p\}$ onto $D-X$.

Let $k=n-m$, and let j be the natural inclusion of $B^m \times R^k$ into S^n. Define f on $(D-X)\times R^k$ by

$$f(x, t) = gj(g^{-1}(x), t), \ x \in D-X, \ t \in R^k.$$

f is an embedding of $(D-X)\times R^k$ into S^n-X which satisfies $f(x, 0)=x$ for $x \in D-X$, and $f((\text{Bd } D-X)\times R^k)$ is locally flat in S^n. The n-cell $\phi(B^n)$ will be constructed in $f(\text{Int } D \times R^k) \cup \text{Bd } D$.

For each x in Int D, let $\varepsilon(x)>0$ be chosen so that $f(x \times B_x)$ has diameter less than the distance from x to Bd D, where B_x is the closed ball in R^k with center 0 and radius $\varepsilon(x)$. $\varepsilon(x)$ may be chosen so that ε is continuous on Int D and so that $\varepsilon(x)=0$, $x \in \text{Bd } D$ defines a continuous extension of ε over all of D. Let

$$N = \{(x, t) \in D \times R^k : \|t\| \leqq \varepsilon(x)\}$$

and $N_0=(\text{Int } D \times R^k) \cap N$. $f|N_0$ can be extended to an embedding $F: N \to S^n$ by letting $F(x, t)=f(x, t)$ if $(x, t) \in N_0$ and $f(x, 0)=x$ if $x \in \text{Bd } D$. It is clear from the construction of N that F is continuous and one-to-one.

The embedding F takes $D \times 0$ onto D. Hence the proof of Theorem 4.1 will be complete as soon as we have shown that $(N, D\times 0) \approx (B^n, B^m)$ and that $F(\text{Bd } N)-X$ is locally flat in S^n.

To prove the first of these assertions, let ε_1 be the continuous function which assigns to each point x of B^m the radius of the ball $B^n \cap H_x$, where H_x is the k-plane in R^n orthogonal to R^m and passing through x. Then

$$B^n = \{(x, t) \in B^m \times R^k : \|t\| \leqq \varepsilon_1(x)\}.$$

If h is a homeomorphism of D onto B^m, h can be extended to a homeomorphism $H: N \approx B^n$ by

$$H(x, t) = (h(x), \varepsilon_1(x)t/\varepsilon(x)), \qquad x \in \text{Int } D$$
$$= h(x), \qquad x \in \text{Bd } D.$$

To see that $F(\text{Bd } N)$ is locally flat at a point not in Bd D, an argument similar to the one in the preceding paragraph will suffice. If x is a point of Bd $D-X$, then the homeomorphism F^{-1} can be extended to a homeomorphism of a neighborhood of x in S^n in the following way: first extend over a neighborhood in $f((D-X)\times R^k)$ by f^{-1}, and then extend over a neighborhood in S^n using the local flatness of $f((\text{Bd } D-X)\times R^k)$. (Actually, there is no range for this last extension to map into. However, $D\times R^k$ may be thought of as being embedded in R^n in such a way that Bd $D\times R^k$ is locally flat.) Thus $F(\text{Bd } N)-X$ is locally flat in S^n, and the theorem is established.

REMARK. The above theorem can be thought of in two ways. First, it gives a way to construct higher dimensional wild cells from lower dimensional ones; and second, it provides a method of taming lower-dimensional cells by knowing that a top-dimensional cell is tame. It is the second application which is used in the following theorem.

THEOREM 4.2. *Let D be an m-cell in S^n, let X be a k-polyhedron in* Bd D, $n \geqq 2k+4$, *and assume that the following conditions hold*:

(1) *$D-X$ is locally flat in S^n*,

(2) *X is locally tame in S^n, and*

(3) *X is locally tame in* Bd D.

Then D is flat in S^n.

Proof. The proof is divided into two cases. First, the theorem is proved assuming that D is a top-dimensional cell. In the second case, Theorem 4.1 is used to "fatten up" a lower-dimensional cell into an n-cell.

Case 1. $m=n$. Thus D is an n-cell in S^n whose boundary is locally flat at each point except possibly at points in X. Let D_1 be an n-cell in Int D such that $D-\text{Int } D_1$ is an n-annulus. Let $Q=(S^n-D)^-$ and $Q_1=(S^n-D_1)^-$. Q_1 is an n-cell by [3]. In order to show that Q is an n-cell (which is equivalent to showing that D is flat) we will construct mappings ϕ of Q_1 onto Q and ψ of Q_1 onto itself. ϕ and ψ will have precisely the same nondegenerate inverse sets, so that the composition $\phi\psi^{-1}$ will be a homeomorphism of Q_1 onto Q.

Construction of ϕ. Let F be a homeomorphism of $D-\text{Int } D_1$ onto $S^{n-1}\times[0, 1]$ such that $F(\text{Bd } D_1)=S^{n-1}\times 1$. It follows from Theorem 1 of [4] that F can be extended to an embedding of $U\cup X$ into $S^{n-1}\times[-1, 2]$, where U is an open set in S^n containing $D-\text{Int } D_1-X$. (We denote the extension by F.) Also, by Theorem 1.1 of [12] and assumption (3), we may assume that $F(X)$ is piecewise linearly embedded in $S^{n-1}\times 0$. Choose a complex K, linearly embedded in S^{n-1}, such that $K\times 0=F(X)$, and let $f=F^{-1}\mid K\times[0, 1]$. Let V be an open set in $S^{n-1}\times[-1, 2]$ such that $(S^{n-1}\times[0, 1]-F(X))\subset V\subset F(U)$ and such that $\bar{V}-F(X)\subset F(U)-F(X)$. Clearly there is a mapping $\tilde{\phi}$ of $S^{n-1}\times[-1, 2]$ onto itself such that $\tilde{\phi}=$identity outside of V and $\tilde{\phi}(S^{n-1}\times 1)=S^{n-1}\times 0$, and such that the nondegenerate inverse sets of $\tilde{\phi}$ are precisely the sets $x\times[0, 1]$, $x\in K$. Define ϕ on Q_1 by

$$\begin{aligned}\phi &= F^{-1}\tilde{\varphi}F \text{ on } U\cap Q_1\\ &= \text{identity on } Q_1-U.\end{aligned}$$

ϕ is a mapping of Q_1 onto Q whose nondegenerate inverse sets are precisely the sets $f(x\times[0, 1])$, $x\in K$.

Construction of ψ. Let G be a homeomorphism of Q_1 onto I^n. (Here $I^1=[0, 1]$ and $I^n=I^{n-1}\times I^1$.) Again applying Theorem 1.1 of [12], we may assume that $Gf\mid K\times 1$ is a piecewise linear embedding of $K\times 1$ into the interior of $I^{n-1}\times 1\subset I^n$. The embedding Gf is clearly locally tame on $K\times(0, 1)$, and $Gf(K\times 0)$ is locally tame in I^n by condition (2). Therefore Gf is a locally tame embedding of $K\times[0, 1)$ into Int I^n by Theorem 1 of [5]. Define $g\colon K\times[0, 1]\to I^n$ by

$$g(x, t) = (Gf(x, 1), (t+1)/2),\ \ x\in K,\ t\in[0, 1].$$

g is a locally tame embedding which agrees with Gf on $K\times 1$. Applying Corollary 3.2, there is a homeomorphism H of I^n onto itself such that $HGf=g$. But clearly there is a mapping $\tilde{\psi}$ of I^n onto itself whose nondegenerate inverse sets are precisely the sets $g(x\times[0, 1])$, $x\in K$, so define ψ on Q_1 by $\psi=G^{-1}H^{-1}\tilde{\psi}HG$. ψ is a mapping of Q_1 onto itself whose nondegenerate inverse sets are precisely the sets $f(x\times[0, 1])$, $x\in K$.

Case 2. $m<n$. Suppose temporarily that X is cellular in both S^n and Bd D. Then, by Theorem 4.1, there is an n-cell $\tilde{D}$ in S^n such that $D\subset\tilde{D}$, $(\tilde{D}, D)\approx(B^n, B^m)$, and Bd $\tilde{D}-X$ is locally flat in S^n. But $\tilde{D}$ is then flat by Case 1, so that the homeomorphism $(\tilde{D}, D)\approx(B^n, B^m)$ can be extended to one of S^n onto itself. (Clearly X is locally tame in Bd $\tilde{D}$ since Bd D is locally tame in Bd $\tilde{D}$.) Thus the theorem is established in the special case in which X is cellular in both Bd D and S^n.

Consider now the general case with no restrictions on X other than local tameness. Let x be a point of X. Since X is locally tame in Bd D, there is a neighborhood V of x in D and a triangulation of V as a combinatorial manifold which contains $V\cap X$ as a subcomplex and x as a vertex. Let (R, R_0) be the closed star of x in the second barycentric subdivision of $(V, V\cap X)$. Then R is an m-cell and R is locally flat in S^n except possibly at the points of $R\cap X=R_0\subset$Bd R. Moreover, R_0 is tame and cellular in both Bd R and S^n. (R_0 is cellular because it is a tame collapsible polyhedron.) It follows that R is flat in S^n and hence that D is locally flat at x. Therefore D is locally flat at every point and must be flat.

COROLLARY 4.3. *Let D be a cell in S^n and E a k-cell in* Bd D, $n \geqq 2k+4$. *If $D-E$ is locally flat in S^n, E is locally flat in S^n, and E is locally flat in* Bd D *then D is flat in S^n.*

Proof. Local flatness implies local tameness.

5. **Applications.** The first theorem in this section provides a method for taming cells whose "bad point set" is not polyhedral.

THEOREM 5.1. *Let D be a cell with locally flat interior in S^n, let B denote the set of points of* Bd D *at which D fails to be locally flat, and let B_0 be an open-closed subset of B. If $B_0 \neq \varnothing$ then B_0 cannot be contained in $X \cap Y$, where X is a tame k-polyhedron in* Bd D *and Y is a tame l-polyhedron in S^n*, $n \geqq 2k+4$, $n \geqq 2l+2$.

Proof. Suppose that such X and Y exist. Then, by Theorem 1 of [5], X is tame in S^n.

Let $f: K \approx X$ be an embedding of the complex K into Bd D such that f is piecewise linear with respect to some triangulation of D as a combinatorial ball. Since $f^{-1}(B_0)$ and $f^{-1}(B-B_0)$ are disjoint closed subsets of K, we may assume that $f(K) \cap (B-B_0) = \varnothing$.

Let $C(K)$ denote the cone over K, and extend f to a piecewise linear embedding F of $C(K)$ into D which takes $C(K)-K$ into Int D. Let N be a regular neighborhood of $F(C(K))$ which does not intersect $B-B_0$. N is an m-cell in D, and N is locally flat in S^n except possibly at the points of $N \cap X = \text{Bd } N \cap X = X$. Since X is tame in Bd N and in S^n, N is flat in S^n, and hence D is locally flat at the points of B_0. This is a contradiction.

REMARK. (1) Let D be a cell with locally flat interior in S^n, and let B denote the set of points of Bd D at which D fails to be locally flat. Corollary 2.5 of [14] shows that if $n \geqq 4$ and $B \neq \varnothing$ then B is a perfect set and hence must *contain* a Cantor set. Theorem 5.1 above implies that if $n \geqq 6$ and B *is* a Cantor set then B must be wild in either Bd D or S^n.

(2) The examples of wild cells in [2] can be used to show that the condition that X be locally tame in S^n is necessary in Theorem 4.2. The author does not know whether the condition that X be locally tame in Bd D is necessary.

We conclude this section by interpreting Theorems 4.2 and 4.3 for embeddings of manifolds. Theorem 5.1 has a similar generalization.

THEOREM 5.2. *Let M and N be combinatorial manifolds of dimension m and n, respectively, with $M \subset$* Int N. *Suppose that $M-X$ is locally flat in N, where X is a k-polyhedron in* Bd M, $n \geqq 2k+4$. *If X is locally tame in both* Bd M *and N then M is locally flat in N.*

Theorem 5.2 is proved in a manner similar to the proof of 5.3 below.

THEOREM 5.3. *Let K, M, and N be (topological) manifolds of dimension k, m, and n, respectively, with $K \subset \mathrm{Bd}\, M \subset M \subset \mathrm{Int}\, N$ and $n \geqq 2k+4$. If $M-K$ and K are locally flat in N and K is locally flat in* Bd *M then M is locally flat in N.*

Proof. Let $x \in K$. Since K is locally flat in Bd M, there is an m-cell D in M such that $D \cap \mathrm{Bd}\, M$ is an $(m-1)$-cell containing x as an interior point, $D \cap K$ is a k-cell locally flat in Bd D, and $D-K$ is locally flat in N. D may be chosen small enough so that D lies in an open n-cell U in N. It follows from Corollary 4.3 that D is locally flat in U, and hence that M is locally flat in N at the point x. Thus M is locally flat in N.

6. **Taming a cell at interior points.** Let $\beta(n, m, m-1)$ denote the following conjecture.

CONJECTURE $\beta(n, m, m-1)$. *Let D_1 and D_2 be two flat m-cells in S^n such that $D_1 \cap D_2 = \mathrm{Bd}\, D_1 \cap \mathrm{Bd}\, D_2$ is an $(m-1)$-cell which is locally flat in both* Bd D_1 *and* Bd D_2. *Then $D_1 \cup D_2$ is a flat cell.*

($\beta(n, m, m-1)$ is one in the class of conjectures considered by Cantrell in [8].)

In [10], Černavskii anounces that $\beta(n, m, m-1)$ is true whenever $n \geqq 5$ and $m \neq n-2$. This section extends the results in §4 to the interior of a cell in any dimension for which $\beta(n, m, m-1)$ holds.

DEFINITION. Let X be a locally finite polyhedron topologically embedded in the n-manifold N; l is a nonnegative integer. X is said to be *locally l-tame* in N if there exist a locally finite complex K and a homeomorphism $f\colon K \approx X$ such that the following holds: given a point x of K, there is a neighborhood U of $f(x)$ in N and a homeomorphism $h\colon \bar{U} \approx B^n$ such that $hf|f^{-1}(\bar{U})$ is a piecewise linear embedding of $f^{-1}(\bar{U})$ into B^{n-l}.

This definition seems rather complicated, and a discussion of the relations between different degrees of local tameness and local embeddability is beyond the scope of this paper. However, it is often easy to decide whether or not a particular embedding is locally l-tame for some $l \geqq 1$, and for this reason we use the definition without further discussion. (It is easy but interesting to list the relations between local 0-tameness, local 1-tameness, local 2-tameness, and local flatness in the case of a 2-manifold in a 4-manifold.)

THEOREM 6.1. *Let D be an m-cell in S^n, let X be a k-polyhedron in D, $n \geqq 2k+4$, and assume that the following hold:*

(1) *$D-X$ is locally flat in S^n;*

(2) *X is locally tame in S^n; and*

(3) *X is locally* 1*-tame in D.*

If $\beta(n, m, m-1)$ is true then D is flat in S^n.

Proof. Let x be a point of X which lies in Int D. Since X is locally 1-tame in D, there is a neighborhood U of x in D and a homeomorphism $h: \bar{U} \approx B^m$ such that $h(\bar{U} \cap X)$ is a subcomplex of B^{m-1}. Clearly we may assume that Bd $\bar{U}$ is locally flat in Int D. Let $B_+(B_-)$ be the set of points of B^m whose last coordinates are nonnegative (resp. nonpositive). Then $B_+ \cup B_- = B^m$ and

$$B_+ \cap B_- = \text{Bd}\, B_+ \cap \text{Bd}\, B_- = B^{m-1}.$$

Define $D_1 = h^{-1}(B_+)$ and $D_2 = h^{-1}(B_-)$. Clearly D_1 and D_2 are locally flat in S^n except possibly at the points of $\bar{U} \cap X$. But $\bar{U} \cap X$ is a k-polyhedron which is locally tame in each of Bd D_1, Bd D_2, and S^n. Hence, since $n \geqq 2k+4$, D_1 and D_2 are flat in S^n by Theorem 4.2. Finally, by the assumption that $\beta(n, m, m-1)$ is true, $D_1 \cup D_2 = \bar{U}$ is a flat cell in S^n, and D is locally flat at the point x. Thus Int D is locally flat in S^n.

Now let x be a point of X in Bd D. Since X is locally tame in D, there is a neighborhood V of x in D such that $\bar{V}$ is an m-cell and $X \cap$ Bd $\bar{V}$ is a locally tame polyhedron in Bd $\bar{V}$ and in S^n. We choose V so that $\bar{V} - X$ is locally flat in S^n. But then $\bar{V} - (X \cap \text{Bd}\, \bar{V})$ is locally flat since Int D is locally flat, and $\bar{V}$ is flat in S^n by Theorem 4.2. Thus D is locally flat at x, and D is a flat cell in S^n.

COROLLARY 6.2. *Let D and E be cells in S^n, $E \subset D$, such that* $(D, E) \approx (B^m, B^k)$, $n \geqq 2k+4$. *If $D-E$ and E are locally flat in S^n* (*and if* $\beta(n, m, m-1)$ *is true*) *then D is flat in S^n.*

REMARK. (1) $\beta(n, n-2, n-3)$ is known to be false for $n \geqq 3$. Moreover, the conclusion of Corollary 6.2 is false when $m = n-2$. See [8] and Corollary 2.6 of [14].

(2) The theorems of §6 can be generalized in the same way that §5 generalizes §4.

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